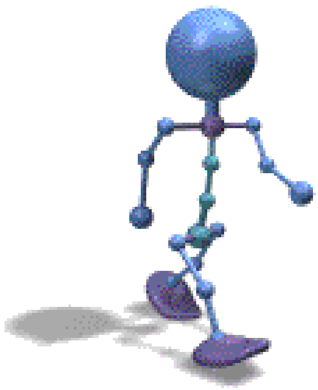
Lundi, 12 Octobre 2015

Optimisation – Licence 3 EF & DU – 2015/2016

Planche de TD N° 2



--- 11H00 → 12H30

Objectifs

Au terme de cette fiche de travaux dirigés, l’étudiant doit être capable de :

1. Reconnaitre l’espace de départ d’une fonction : $\mathbb{R}, \mathbb{R}^2$, etc.
2. Reconnaitre l’espace d’arrivé d’une fonction
3. Calculer la dérivée de fonctions (première, seconde, etc.)
4. Calculer les dérivées partielles de fonctions (première, seconde, etc.)
5. Calculer la dérivée de fonctions composées
6. Calculer les dérivés partielles de fonctions composées
7. Effectuer des comparaisons

$f : \mathbb{R} \to \mathbb{R}$ $t \mapsto f(t)$	$\frac{df}{dt}(t) = \boldsymbol{f'(t)}$
$f : \mathbb{R} \to \mathbb{R}^n$ $t \mapsto \left(f_1(t), \dots, f_n(t)\right)$	$\frac{df}{dt}(t) = \boldsymbol{f'(t)} = \left(f_1'(t), \dots, f_n'(t)\right)$
$f : \mathbb{R}^n \to \mathbb{R}$ $x = (x_1, \dots, x_n) \mapsto f(x)$	$\frac{\partial f}{\partial x_i}(x) = f'_{x_1,\dots,x_{i-1},x_{i+1},\dots,x_n}(x_i)$ $df(x) = \boldsymbol{f'(x)} = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\right)$

$$(fog)' = g' \times f'og$$

Exemple 1

$$f : \mathbb{R}^n \to \mathbb{R}$$
$$x = (x_1, \dots, x_n) \mapsto f(x)$$

$$u : \mathbb{R} \to \mathbb{R}^n$$
$$t \mapsto \big(u_1(t), \dots, u_n(t)\big)$$

$$u_1 : \mathbb{R} \to \mathbb{R}$$
$$\text{t} \mapsto u_1(t)$$

...

$$u_n : \mathbb{R} \to \mathbb{R}$$
$$\text{t} \mapsto u_n(t)$$

$$g : \mathbb{R} \to \mathbb{R}, t \mapsto g(t) = f\big(u_1(t), \dots, u_n(t)\big)$$

$g = f o\, u$ donc $g' = (fou)' = u' \times f' o\, u$

$$u'(t) = \big(u_1'(t), \dots, u_n'(t)\big)$$
$$f'(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\right)$$

$$u(t) = \big(u_1(t), \dots, u_n(t)\big)$$

$$u'(t) = \big(u'_1(t), \dots, u'_n(t)\big)$$

$$g' = (f o u)' = u' \times f' o\, u$$

$$f'(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\right)$$

$$f' o\, u \equiv f'\big(u(t)\big) = \left(\frac{\partial f}{\partial x_1}\big(u(t)\big), \dots, \frac{\partial f}{\partial x_n}\big(u(t)\big)\right)$$

$$= \left(\frac{\partial f}{\partial x_1}\big(u_1(t), \dots, u_n(t)\big), \dots, \frac{\partial f}{\partial x_n}\big(u_1(t), \dots, u_n(t)\big)\right)$$

$$u'(t)(f' o\, u)(t) = \big(u'_1(t), \dots, u'_n(t)\big) \times \left(\frac{\partial f}{\partial x_1}\big(u_1(t), \dots, u_n(t)\big), \dots, \frac{\partial f}{\partial x_n}\big(u_1(t), \dots, u_n(t)\big)\right)$$

$$= u'_1(t) \times \frac{\partial f}{\partial x_1}\big(u_1(t), \dots, u_n(t)\big) + \dots + u'_n(t) \times \frac{\partial f}{\partial x_n}\big(u_1(t), \dots, u_n(t)\big)$$

Exercice 1

$f: \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \mapsto f(x,y,z)$ et $g(t) = t^2 f(3t+2, t^2, e^t)$

g est sous la forme $h \times f o\, u$ avec :

$h(t) = t^2$ et $u : \mathbb{R} \to \mathbb{R}^3$, $t \mapsto \big(u_1(t), u_2(t), u_3(t)\big)$

$u_1: \mathbb{R} \to \mathbb{R}$, $t \mapsto u_1(t)$=3t+2

$u_2: \mathbb{R} \to \mathbb{R}$, $t \mapsto u_2(t)=t^2$

$u_3: \mathbb{R} \to \mathbb{R}$, $t \mapsto u_3(t)=e^t$

$$g = h(f o\, u) \equiv g' = h'(f\ o\ u) + h\,(f\ o\ u)' \quad \text{or} \quad h'(t) = 2t$$

$$(f\ o\ u)' = u_1'(t)\,\frac{\partial f}{\partial x}\big(u_1(t), u_2(t), u_3(t)\big) + u_2'(t)\,\frac{\partial f}{\partial y}\big(u_1(t), u_2(t), u_3(t)\big) + u_3'(t)\,\frac{\partial f}{\partial z}\big(u_1(t), u_2(t), u_3(t)\big)$$

$$= 3\,\frac{\partial f}{\partial x}(3t+2, t^2, e^t) + 2t\,\frac{\partial f}{\partial y}(3t+2, t^2, e^t) + e^t\,\frac{\partial f}{\partial z}(3t+2, t^2, e^t)$$

Exercice 1

$f\,:\,\mathbb{R}^3 \to \mathbb{R}\,,\ (x,y,z) \mapsto f(x,y,z)$ et $g(t) = t^2 f(3t+2, t^2, e^t)$

$$g = h(f\,o\,u) \equiv g' = h' \times (f\,o\,u) + h \times (f\,o\,u)'$$

$$h'(t) = 2t$$

$$(f\,o\,u)(t) = f\big(u(t)\big) = f\big(u_1(t), u_2(t), u_3(t)\big) = f(3t+2, t^2, e^t)$$

$$(f\,o\,u)' = 3\frac{\partial f}{\partial x}(3t+2\,, t^2, e^t) + 2t\frac{\partial f}{\partial y}(3t+2\,, t^2, e^t) + e^t\frac{\partial f}{\partial z}(3t+2\,, t^2, e^t)$$

$$g'(t) = 2tf(3t+2, t^2, e^t) + t^2\left[3\frac{\partial f}{\partial x}(3t+2\,, t^2, e^t) + 2t\frac{\partial f}{\partial y}(-) + e^t\frac{\partial f}{\partial z}(-)\right]$$

$$= 2tf(3t+2, t^2, e^t) + 3t^2\frac{\partial f}{\partial x}(-) + 2t^3\frac{\partial f}{\partial y}(-) + t^2e^t\frac{\partial f}{\partial z}(-)$$

Exemple 2

$$f : \mathbb{R}^n \to \mathbb{R}$$
$$x = (x_1, \dots, x_n) \mapsto f(x)$$

$$u : \mathbb{R}^p \to \mathbb{R}^n$$
$$\mathrm{t} = (t_1, \dots, t_p) \mapsto (u_1(t), \dots, u_n(t))$$

$$u_1 : \mathbb{R}^p \to \mathbb{R}$$
$$\mathrm{t} \mapsto u_1(t)$$

…

$$u_n : \mathbb{R}^p \to \mathbb{R}$$
$$\mathrm{t} \mapsto u_n(t)$$

$$g : \mathbb{R}^p \to \mathbb{R}, \mathrm{t} = (t_1, \dots, t_p) \mapsto g(t) = f(u_1(t), \dots, u_n(t))$$

Calcul de la dérivée partielle de g par rapport à t_i : $\frac{\partial g}{\partial t_i}$

On suppose tout les autres t_k pour $k \neq i$

$$f : \mathbb{R}^n \to \mathbb{R}$$
$$x = (x_1, \dots, x_n) \mapsto f(x)$$

$$u : \mathbb{R} \to \mathbb{R}^n$$
$$t_i \mapsto (u_1(t_i), \dots, u_n(t_i))$$

$$u_1 : \mathbb{R} \to \mathbb{R}$$
$$t_i \mapsto u_1(t)$$

…

$$u_n : \mathbb{R} \to \mathbb{R}$$
$$t_i \mapsto u_n(t)$$

Exemple 2

$$g'(t) = u_1'(t) \times \frac{\partial f}{\partial x_1}\big(u_1(t), \dots, u_n(t)\big) + \dots + u_n'(t) \times \frac{\partial f}{\partial x_n}\big(u_1(t), \dots, u_n(t)\big)$$

$$\text{t} = \big(t_1, \dots, t_i, \dots, t_p\big)$$

$$\frac{\partial g}{\partial t_i}(t) = \frac{\partial u_1}{\partial t_i}(t)\frac{\partial f}{\partial x_1}\big(u_1(t), \dots, u_n(t)\big) + \cdots + \frac{\partial u_n}{\partial t_i}(t)\frac{\partial f}{\partial x_n}\big(u_1(t), \dots, u_n(t)\big)$$

Exercice 2

$f : \mathbb{R}^3 \to \mathbb{R}\ ,\ (x,y,z) \mapsto f(x,y,z)$ et $g(s,t) = (2s+t)f(s^2+t, 2t, 3s+2t)$

g est sous la forme $h \times f o\, u$ avec :

$h: \mathbb{R}^2 \to \mathbb{R}$ et $u : \mathbb{R}^2 \to \mathbb{R}^3$

(s,t) $\mapsto 2s+t$ $(s,t) \mapsto \big(u_1(s,t), u_2(s,t), u_3(s,t)\big)$

$u_1: \mathbb{R}^2 \to \mathbb{R}$, t $\mapsto u_1(t)$=s^2+t

$u_2: \mathbb{R}^2 \to \mathbb{R}$, t $\mapsto u_2(t)$=2t

$u_3: \mathbb{R}^2 \to \mathbb{R}$, t $\mapsto u_3(t)$=3s+2t

$g = h(f o\, u) \equiv g' = h'(f\ o\ u) + h\,(f\ o\ u)'$

t= $\left(t_1, \dots, t_i, \dots, t_p\right)$

$$\frac{\partial (f o\, u)}{\partial t_i}(t) = \frac{\partial u_1}{\partial t_i}(t)\frac{\partial f}{\partial x_1}\big(u_1(t), \dots, u_n(t)\big) + \cdots + \frac{\partial u_n}{\partial t_i}(t)\frac{\partial f}{\partial x_n}\big(u_1(t), \dots, u_n(t)\big)$$

$$\frac{\partial (f o\, u)}{\partial s}(s,t) = \frac{\partial u_1}{\partial s}(s,t)\frac{\partial f}{\partial x}\big(u_1(s,t), u_2(s,t), u_3(s,t)\big) + \frac{\partial u_2}{\partial s}(s,t)\frac{\partial f}{\partial y}(-) + \frac{\partial u_3}{\partial s}(s,t)\frac{\partial f}{\partial z}(-)$$

$$= 2s\frac{\partial f}{\partial x}(s^2+t\,, 2t, 3S+2t\ ) + 0 + 3\frac{\partial f}{\partial z}(-)$$

$$g = h(f o\, u) \equiv g' = h'(f\ o\ u) + h\,(f\ o\ u)'$$

$$\frac{\partial g}{\partial s}(s,t) = \frac{\partial h}{\partial s}(s,t) \times (f\ o\ u)(s,t) + h(s,t) \times \frac{\partial (f o\, u)}{\partial s}(s,t)$$

$h(s,t) = 2s + t$ donc $\quad \frac{\partial h}{\partial s}(s,t) = 2$

$$(f\ o\ u)(s,t) = f(s^2 + t, 2t, 3s + 2t)$$

$$\frac{\partial (f o\, u)}{\partial s}(s,t) = 2s\frac{\partial f}{\partial x}(s^2 + t\,, 2t, 3S + 2t\,) + 0 + 3\frac{\partial f}{\partial z}(-)$$

$$\frac{\partial g}{\partial s}(s,t) = 2f(s^2 + t, 2t, 3s + 2t) + (2s + t)\left[2s\frac{\partial f}{\partial x}(s^2 + t\,, 2t, 3S + 2t\,) + 0 + 3\frac{\partial f}{\partial z}(-)\right]$$

$$= 2f(s^2 + t, 2t, 3s + 2t) + 2s\,(2s + t)\frac{\partial f}{\partial x}(-) + 3(2s + t)\frac{\partial f}{\partial z}(-)$$

$$\frac{\partial g}{\partial t}(s,t) = \frac{\partial h}{\partial t}(s,t)\, f(s^2\text{+t, 2t, 3s+2t})+h(s,t)\frac{\partial}{\partial t}\left(f(s^2\text{+t, 2t, 3s+2t})\right)$$

$$= f(s^2\text{+t, 2t, 3s+2t})+ (2s+t)\left[\frac{\partial f}{\partial x}(-) + 2\frac{\partial f}{\partial y}(-) + 2\frac{\partial f}{\partial z}(-)\right]$$

$$\boldsymbol{\frac{\partial g}{\partial t}(s,t) = f(s^2\text{+t, 2t, 3s+2t})+ (2s+t)\frac{\partial f}{\partial x}(-) + 2(2s+t)\frac{\partial f}{\partial y}(-) + 2(2s+t)\frac{\partial f}{\partial z}(-)}$$

Exercice 2

Calcul des dérivées partielles secondes :

$$\frac{\partial g}{\partial s}(s,t) = 2\,f(s^2+t,\ 2t,\ 3s+2t) + 2s\,(2s+t)\frac{\partial f}{\partial x}(-) + 3(2s+t)\frac{\partial f}{\partial z}(-)$$

$$\frac{\partial g}{\partial t}(s,t) = f(s^2+t,\ 2t,\ 3s+2t) + (2s+t)\frac{\partial f}{\partial x}(-) + 2(2s+t)\frac{\partial f}{\partial y}(-) + 2(2s+t)\frac{\partial f}{\partial z}(-)$$

$$\frac{\partial^2 g}{\partial s^2}(s,t) = \frac{\partial}{\partial s}\left[2\,f(s^2+t,\ 2t,\ 3s+2t)\right]$$

$$+ \quad \frac{\partial}{\partial s}\left[2s\,(2s+t)\frac{\partial f}{\partial x}(s^2+t,\ 2t,\ 3s+2t)\right]$$

$$+ \quad \frac{\partial}{\partial s}\left[3(2s+t)\frac{\partial f}{\partial z}(s^2+t,\ 2t,\ 3s+2t)\right]$$

$$\frac{\partial}{\partial s}[2\,f(s^2\text{+t, 2t, 3s+2t})] = 2\left[2S\frac{\partial f}{\partial x}(-) + 0\times\frac{\partial f}{\partial y}(-) + 3\frac{\partial f}{\partial z}(-)\right]$$

$$= 4S\frac{\partial f}{\partial x}(-) + 6\frac{\partial f}{\partial z}(-)$$

$$\frac{\partial}{\partial s}\left[2s\,(2s+t)\frac{\partial f}{\partial x}\,(s^2\text{+t, 2t, 3s+2t})\right] = \frac{\partial}{\partial s}\left[\,(4s^2+2st)\frac{\partial f}{\partial x}\,(s^2\text{+t, 2t, 3s+2t})\right]$$

$$= (8s+2t)\frac{\partial f}{\partial x}\,(s^2\text{+t, 2t, 3s+2t})+2s\,(2s+t)\left[2s\frac{\partial^2 f}{\partial x^2}(-) + 0\times\frac{\partial^2 f}{\partial y\partial x}(-) + 3\frac{\partial^2 f}{\partial z\partial x}(-)\right]$$

$$= (8s+2t)\frac{\partial f}{\partial x}\,(s^2\text{+t, 2t, 3s+2t})+4s^2(2s+t)\frac{\partial^2 f}{\partial x^2}(-) + 6s(2s+t)\frac{\partial^2 f}{\partial z\partial x}(-)$$

$$\frac{\partial}{\partial s}\left[3(2s+t)\frac{\partial f}{\partial z}(s^2\text{+t, 2t, 3s+2t})\right] = \frac{\partial}{\partial s}\left[(6s+3t)\frac{\partial f}{\partial z}(s^2\text{+t, 2t, 3s+2t})\right]$$

$$= 6\frac{\partial f}{\partial z}(s^2\text{+t, 2t, 3s+2t})+\,3(2s+t)\left[2s\frac{\partial^2 f}{\partial x\partial z}(-) + 0\times\frac{\partial^2 f}{\partial y\partial z}(-) + 3\frac{\partial^2 f}{\partial z^2}(-)\right]$$

$$= 6\frac{\partial f}{\partial z}(s^2\text{+t, 2t, 3s+2t})+\,6s(2s+t)\frac{\partial^2 f}{\partial x\partial z}(-) + 9(2s+t)\frac{\partial^2 f}{\partial z^2}(-)$$

Exercice 2

Calcul des dérivées partielles secondes :

$$\frac{\partial g}{\partial s}(s,t) = 2\,f(s^2+t,\,2t,\,3s+2t) + 2s\,(2s+t)\frac{\partial f}{\partial x}(-) + 3(2s+t)\frac{\partial f}{\partial z}(-)$$

$$\frac{\partial g}{\partial t}(s,t) = f(s^2+t,\,2t,\,3s+2t) + (2s+t)\frac{\partial f}{\partial x}(-) + 2(2s+t)\frac{\partial f}{\partial y}(-) + 2(2s+t)\frac{\partial f}{\partial z}(-)$$

$$\frac{\partial^2 g}{\partial t \partial s}(s,t) = \frac{\partial}{\partial t}\left[2\,f(s^2+t,\,2t,\,3s+2t)\right]$$

$$+ \quad \frac{\partial}{\partial t}\left[2s\,(2s+t)\frac{\partial f}{\partial x}\,(s^2+t,\,2t,\,3s+2t)\right]$$

$$+ \quad \frac{\partial}{\partial t}\left[3(2s+t)\frac{\partial f}{\partial z}(s^2+t,\,2t,\,3s+2t)\right]$$

$$\frac{\partial}{\partial t}[2\,f(s^2\text{+t, 2t, 3s+2t})] = 2\left[\frac{\partial f}{\partial x}(-) + 2\frac{\partial f}{\partial y}(-) + 2\frac{\partial f}{\partial z}(-)\right]$$

$$= 2\frac{\partial f}{\partial x}(-) + 4\frac{\partial f}{\partial y}(-) + 4\frac{\partial f}{\partial z}(-)$$

$$\frac{\partial}{\partial t}\left[2s\,(2s+t)\frac{\partial f}{\partial x}\,(s^2\text{+t, 2t, 3s+2t})\right] = \frac{\partial}{\partial t}\left[\,(4s^2+2st)\frac{\partial f}{\partial x}\,(s^2\text{+t, 2t, 3s+2t})\right]$$

$$= 2s\frac{\partial f}{\partial x}\,(s^2\text{+t, 2t, 3s+2t}) + 2s\,(2s+t)\left[\frac{\partial^2 f}{\partial x^2}(-) + 2\frac{\partial^2 f}{\partial y \partial x}(-) + 2\frac{\partial^2 f}{\partial z \partial x}(-)\right]$$

$$= 2s\frac{\partial f}{\partial x}\,(s^2\text{+t, 2t, 3s+2t}) + 2s(2s+t)\frac{\partial^2 f}{\partial x^2}(-) + 4s(2s+t)\frac{\partial^2 f}{\partial y \partial x}(-) + 4s(2s+t)\frac{\partial^2 f}{\partial z \partial x}(-)$$

$$\frac{\partial}{\partial t}\left[3(2s+t)\frac{\partial f}{\partial z}(s^2\text{+t, 2t, 3s+2t})\right] = \frac{\partial}{\partial t}\left[(6s+3t)\frac{\partial f}{\partial z}(s^2\text{+t, 2t, 3s+2t})\right]$$

$$= 3\frac{\partial f}{\partial z}(s^2\text{+t, 2t, 3s+2t}) + 3(2s+t)\left[\frac{\partial^2 f}{\partial x \partial z}(-) + 2\frac{\partial^2 f}{\partial y \partial z}(-) + 2\frac{\partial^2 f}{\partial z^2}(-)\right]$$

$$= 3\frac{\partial f}{\partial z}(s^2\text{+t, 2t, 3s+2t}) + 3s(2s+t)\frac{\partial^2 f}{\partial x \partial z}(-) + 6s(2s+t)\frac{\partial^2 f}{\partial y \partial z}(-) + 6(2s+t)\frac{\partial^2 f}{\partial z^2}(-)$$

Exercice 3

$$g(x) = x^2 f(3x+2, x^2, e^x)$$

Dérivée de g en fonction des dérivées partielles de f

$$g'(x) = 2xf(3x+2, x^2, e^x) + x^2\left[3\frac{\partial f}{\partial x}(-) \ + 2x\frac{\partial f}{\partial y}(-) + e^x\frac{\partial f}{\partial z}(-)\right]$$

$$= 2xf(3x+2, x^2, e^x) + 3x^2\frac{\partial f}{\partial x}(-) \ + 2x^3\frac{\partial f}{\partial y}(-) + x^2e^x\frac{\partial f}{\partial z}(-)$$

Exercice 4

$$g(x,y) = (2x+y)f(x^2+y, 2y, 3\text{x}+2\text{y})$$

Dérivées partielles de g en fonction des dérivées partielles de f

$$\frac{\partial g}{\partial x}(x,y) = 2f(x^2+y, 2y, 3\text{x}+2\text{y}) + (2x+y)\left[2x\frac{\partial f}{\partial x}(-) + 0\times\frac{\partial f}{\partial y}(-) + 3\frac{\partial f}{\partial z}(-)\right]$$

$$= 2f(x^2+y, 2y, 3\text{x}+2\text{y}) + 2x(2x+y)\frac{\partial f}{\partial x}(-) + 3(2x+y)\frac{\partial f}{\partial z}(-)$$

$$\frac{\partial g}{\partial y}(x,y) = f(x^2+y, 2y, 3\text{x}+2\text{y}) + (2x+y)\left[\frac{\partial f}{\partial x}(-) + 2\frac{\partial f}{\partial y}(-) + 2\frac{\partial f}{\partial z}(-)\right]$$

$$= f(x^2+y, 2y, 3\text{x}+2\text{y}) + (2x+y)\frac{\partial f}{\partial x}(-) + 2(2x+y)\frac{\partial f}{\partial y}(-) + 2(2x+y)\frac{\partial f}{\partial z}(-)$$

Exercice 5

$$f(x, y, z) = x + 2yz^3$$

Calcul des dérivées partielles premières de f

$$\frac{\partial f}{\partial x}(x, y, z) = 1; \quad \frac{\partial f}{\partial y}(x, y, z) = 2z^3 \quad \text{et} \quad \frac{\partial f}{\partial z}(x, y, z) = 6yz^2$$

$$g(x, y) = f(x, y, x)$$

Calculer $g(x, y)$

$$g(x, y) = f(x, y, x) = x + 2yx^3$$

Calcul des dérivées partielles premières de g

$g(x,y) = x + 2yx^3, \frac{\partial f}{\partial x}(x,y,z) = 1;$ $\frac{\partial f}{\partial y}(x,y,z) = 2z^3$ et $\frac{\partial f}{\partial z}(x,y,z) = 6yz^2$

$\frac{\partial g}{\partial x}(x,y) = 1 + 6x^2y$ et $\frac{\partial g}{\partial y}(x,y) = 2x^3$

Comparaison

$$g(x,y) = f(x,y,x)$$

$$\frac{\partial g}{\partial x}(x,y) = \frac{\partial f}{\partial x}(x,y,x) + 0 \times \frac{\partial f}{\partial y}(x,y,x) + \frac{\partial f}{\partial z}(x,y,x)$$

$= \frac{\partial f}{\partial x}(x,y,x) + \frac{\partial f}{\partial z}(x,y,x)$ or $\frac{\partial f}{\partial z}(x,y,x) = 6yx^2$

$$\frac{\partial g}{\partial x}(x,y) = \frac{\partial f}{\partial x}(x,y,x) + 6yx^2$$

Fin

Merci Pour Votre Attention