# Série 4

## Exercice ~~4~~ 1

$X \longrightarrow f(\cdot) \qquad (x_1, x_2, x_3)$

$E(X) = m \qquad V(X = \sigma^2)$

$n = 3$

$T_1 = \dfrac{X_1 + X_2 + X_3}{3} \qquad T_2 = \dfrac{X_1 + 2X_2 + X_3}{4}$

1, Sans biais

$E(T_1) = m \qquad T_1$ sans biais

$E(T_1) = E\left(\dfrac{X_1 + X_2 + X_3}{3}\right) = \dfrac{E(X_1) + E(X_2) + E(X_3)}{3} = \dfrac{3m}{3} = m$

$E(T_2) = \dfrac{E(X_1 + 2X_2 + X_3)}{4} = \dfrac{4m}{4} = m$

2, $V(T_1) = V\left(\dfrac{X_1 + X_2 + X_3}{3}\right)$

$= \dfrac{1}{9}\left[V(X_1) + V(X_2) + V(X_3)\right] = \dfrac{1}{9} \cdot 3\sigma^2 = \dfrac{\sigma^2}{3}$

$V(T_2) = \dfrac{V(X_1 + 2X_2 + X_3)}{4} = \dfrac{1}{16}\left[V(X_1) + 4V(X_2) + V(X_3)\right]$

$= \dfrac{6\sigma^2}{16} = \dfrac{3}{8}\sigma^2$

$V(T_1) < V(T_2) \Rightarrow T_1$ le ~~mil~~ meilleur

3) $T_3 = ax_1 + bx_2 + cx_3$

$E(T_3) = aE(X_1) + bE(X_2) + cE(X_3)$

$= (a + b + c)m$

$T_3$ sans biais $\Rightarrow E(T_3) = m \Rightarrow a + b + c = 1$

$V(T_3) = V(aX_1 + bX_2 + cX_3)$

$= a^2V(X_1) + b^2V(X_2) + c^2V(X_3)$

$= (a^2 + b^2 + c^2)\sigma^2$

. Conditions

$$\min_{(a,b,c)} V(T_3) = (a^2+b^2+c^2)\sigma^2$$
$$a+b+c=1$$

$$L(a,b,c,\lambda) = (a^2+b^2+c^2)\sigma^2 - \lambda(a+b+c-1)$$

. FOC $= \frac{\partial L}{\partial a} = 0 \Rightarrow 2a\sigma^2 - \lambda = 0 \Rightarrow a = \frac{\lambda}{2\sigma^2}$

$\frac{\partial L}{\partial b} = 0 \Rightarrow 2b\sigma^2 - \lambda = 0 \Rightarrow b = \frac{\lambda}{2\sigma^2}$

$\frac{\partial L}{\partial c} = 0 \Rightarrow 2c\sigma^2 - \lambda = 0 \Rightarrow c = \frac{\lambda}{2\sigma^2}$

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow a+b+c-1 = 0 \Rightarrow a=b=c=\frac{1}{3}$

Exercice 2

$$y_i = \begin{cases} 1 & p \\ 0 & q = 1-p \end{cases}$$

1, $P(y_i = k_i) = p^{k_i}(1-p)$ pour $k_i = 0$ ou $1$

$P(y_i = 0) = p^0 \cdot (1-p)^{1-0} = 1-p$

$P(y_i = 1) = p^1 \cdot (1-p)^{1-1} = p$

$y_i \sim$ Bernouilli $(p)$

$X = \sum y_i$

$L(y_1, y_2, \dots, y_n) \equiv P(y_1, y_2, \dots y_n)$

$= P(y_1) \cdot P(y_2) \cdot P(y_3) \dots P(y_n)$

$= p^{k_1}(1-p)^{1-k_1} \cdot p^{k_2}(1-p)^{1-k_2} \dots p^{k_n}(1-p)^{1-k_n}$

$= p^{\sum k_i} \cdot (1-p)^{\sum(1-k_i)}$

$= p^{\sum k_i}(1-p)^n \quad k = \sum k_i \quad E(k) = np, \; E(n-k) = n(1-p)$

$\max p^k (1-p)^{n-k}$

$\max \log[p^k(1-p)^{n-k}]$

$\max k \cdot \log p + (n-k)\log(1-p)$

FOC $\frac{\partial L}{\partial p} = k \cdot \frac{1}{p} + (n-k) \cdot \frac{1}{1-p}(-1) = 0$

$\Rightarrow \frac{k}{p} = \frac{n-k}{1-p} \Rightarrow p(n-k) = k(1-p)$

$\Rightarrow pn - pk = k - kp$

$\Rightarrow pn = k$

$\Rightarrow \cancel{pn}\ p = \frac{k}{n} = \frac{\sum ki}{n}$

$F = \frac{\sum ki}{n}$

$$\frac{\partial^2 L}{\partial p^2} = \frac{-k}{p^2} - \frac{(n-k)}{(1-p)^2}$$

$$= \frac{(\sum ki)}{p^2} - \frac{(n - \sum_{i=1}^{n} ki)}{(1-p)^2}$$

$$E\left(-\frac{\partial^2 L}{\partial p^2}\right) = E\left(\frac{k}{p^2} + \frac{n-k}{(1-p)^2}\right)$$

$$= \frac{E(k)}{p^2} + \frac{E(n-k)}{(1-p)^2} = \frac{np}{p^2} + \frac{n(1-p)}{(1-p)^2}$$

$$I = E\left(-\frac{\partial^2 L}{\partial p^2}\right) = \frac{n}{p} + \frac{n}{(1-p)}$$

$$= n\left[\frac{p+(1-p)}{p(1-p)}\right] = \frac{n}{p(1-p)}$$

I informat° matrix

$$V(F) = \frac{1}{I(F)} = \frac{p(1-p)}{n} = \frac{pq}{n}$$

Exercice 3

$(y_1, y_2, \ldots, y_n)$ échantillons aléatoire) $y = \sum y_i \rightsquigarrow$ Binomial $(n, p)$

$n = 700$ $p = ?$

$E(y) = np$ $V(y) = npq$

$E(F) = p$ $V(F) = \frac{pq}{n}$

1, $F = \frac{\sum_{i=1}^{700} y_i}{700}$

Il a raison car c'est 1 estimateur sans biais convergent et efficace

On obtient comme estimateur de p

$$F = \frac{\sum_{i=1}^{700} y_i}{700} = \frac{364}{700} = 52\%$$

2) $n = 700 \quad p = 0,52 \quad np > 30 , nq > 30$

$$TCL \Rightarrow \frac{F-p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1)$$

$P(a \leqslant F \leqslant b) = 1-\alpha = 98\% \quad a? \quad b?$

$$P\left(\frac{a-p}{\sqrt{\frac{p(1-p)}{n}}} \leqslant \frac{F-p}{\sqrt{\frac{p(1-p)}{n}}} \leqslant \frac{b-p}{\sqrt{\frac{p(1-p)}{n}}}\right)$$

$$= P\left(\frac{a-p}{\sqrt{\frac{p(1-p)}{n}}} \leqslant U \leqslant \frac{b-p}{\sqrt{\frac{p(1-p)}{n}}}\right) = 98\%$$

Symétric : $a-p = -(b-p)$

$$= P\left[\frac{-(b-p)}{\sqrt{\frac{p(1-p)}{n}}} \leqslant U \leqslant \frac{b-p}{\sqrt{\frac{p(1-p)}{n}}}\right.$$

$$\phi\left(\frac{b-p}{\sqrt{\frac{p(1-p)}{n}}}\right) - \phi\left(\frac{-(b-p)}{\sqrt{\frac{p(1-p)}{n}}}\right)$$

$$= 2\phi\left(\frac{b-p}{\sqrt{\frac{p(1-p)}{n}}}\right) - 1 = 0,98$$

$$= \phi\left(\frac{b-p}{\sqrt{\frac{p(1-p)}{n}}}\right) = \frac{1,98}{2} = 0,99$$

Table 2

$$\frac{b-p}{\sqrt{\frac{p(1-p)}{n}}} = 2,33$$

$$b = p + 2,33\sqrt{\frac{p(1-p)}{n}}$$

$p = 52\% \quad n = 700$

$$b = 0,52 + 2,33\sqrt{\frac{0,52 \times 0,48}{700}} = 0,52 + 0,044 = 0,56$$

$$a = p - 2,33\sqrt{\frac{p(1-p)}{n}} = 0,52 - 0,33 \times \sqrt{\frac{0,48 \times 0,52}{700}} = 0,476$$