Série 3

Exercice 1

1) Soit X_i loi variable erreur associate a un nombre i.
X_i suit une loi uniforme sur $[-0,5 \, ; 0,5]$

$$X \rightsquigarrow U(a,b)$$

$$E(x) = \frac{b+a}{2} \qquad V(x) = \frac{(b-a)^2}{12}$$

$a = -0,5 \qquad b = 0,5$

$$E(X_i) = \frac{0,5+(-0,5)}{2} = 0$$

$$S_{1500} = \sum_{i=1}^{1500} X_i \Rightarrow E(S_{1500}) = 0$$

$$V(S_{1500}) = V(\Sigma X_i) = \sum_{i=1}^{1500} V(X_i)$$

$$= 1500 \cdot \frac{1}{12} = 125$$

Par le TCL, on a $\frac{S_{1500}-0}{\sqrt{125}} \sim N(0,1)$

$$P(|S_{1500}| > 15) = P\left(\left|\frac{S_{1500}-0}{\sqrt{125}}\right| > \frac{15-0}{\sqrt{125}}\right)$$

$$= P\left(|U| > \frac{15}{5\sqrt{5}}\right) = P(|U| > 1,34)$$

$$= P(U > 1,34) + P(U < -1,34)$$

$$= 1 - \phi(1,34) + 1 - \phi(1,34)$$

$$= 2 - 2\phi(1,34)$$

$$= 2 - 2 \times 0,9099 = 0,18$$

b, On cherche n tel que $|S_n| < 10$ avec

$$(P(|S_n|) < 10) = 90\%$$

$$E(S_n) = E\left(\sum_{i=1}^{n} x_i\right) = \Sigma E(x_i) = n \cdot 0 = 0$$

$$V(S_n) = V(\Sigma x_i) = \Sigma V(x_i) = n \cdot \frac{1}{12}$$

$$\frac{S_n - E(S_n)}{\sqrt{V(S_n)}} \rightsquigarrow N(0,1) \Rightarrow P(|S_n| < 10)$$

$$= P\left(\left|\frac{S_n - 0}{\sqrt{\frac{n}{12}}}\right| < \frac{10-0}{\sqrt{\frac{n}{12}}}\right)$$

$$= P\left(|U| < \frac{20\sqrt{3}}{\sqrt{n}}\right) = 90\%$$

Parce que $P(|U| < 1,645) = 90\%$

$$\frac{20\sqrt{3}}{\sqrt{n}} = 1,645 \qquad n = \left(\frac{20\sqrt{3}}{1,645}\right) = 444$$

Exercice 2

1, Un individu i de l'échantillon, on peut associer une variable de Bernouilli de paramètre π

$$X_i = \begin{cases} 1 & \pi \\ 0 & (1-\pi) \end{cases}$$

$$P = \frac{\sum_{i=1}^{500} X_i}{500} \Rightarrow E(P) = \frac{\Sigma(X_i)}{500} = \frac{500\pi}{500} = \pi$$

$$V(P) = \frac{\Sigma V(X_i)}{500^2} = \frac{500\, V(X_i)}{500^2} = \frac{\pi(1-\pi)}{500}$$

$$\sigma(p) = \sqrt{\frac{\pi(1-\pi)}{500}} = \sqrt{\frac{\pi(1-\pi)}{10\sqrt{5}}}$$

Par le TCL, on a $\dfrac{P-\pi}{\frac{\sqrt{\pi(1-\pi)}}{10\sqrt{5}}} \sim N(0,1)$

$$P(P < 1/2) = P\left[\frac{P-\pi}{\frac{\sqrt{\pi(1-\pi)}}{10\sqrt{5}}} < \frac{1/2-\pi}{\frac{\sqrt{\pi(1-\pi)}}{10\sqrt{5}}}\right]$$

$$= P\left(u < \frac{(1/2-\pi)\, 10\sqrt{5}}{\sqrt{\pi(1-\pi)}}\right)$$

$$= P(u < -0,895) = 18,54\%$$

2, On a a) $n = 500$

b) $n = 2000$

$$P = \frac{\sum_{i=1}^{n} X_i}{n} \qquad E(P) = \frac{E(\sum X_i)}{n} = \frac{n\pi}{n} = \pi = 52\%$$

$$V(p) = \frac{V(\sum X_i)}{n^2} = \frac{n\,V(X_i)}{n^2} = \frac{\pi(1-\pi)}{n}$$

$$\sigma(p) = \sqrt{\frac{\pi(1-\pi)}{n}}$$

Par TCL :

$$P(P < \frac{1}{2}) = P\left(\frac{P-\pi}{\sqrt{\frac{\pi(1-\pi)}{2000}}} < \frac{1/2 - \pi}{\sqrt{\frac{\pi(1-\pi)}{2000}}}\right)$$

$$= P(U < -1{,}75) = 3{,}7\%$$

c, $n = 8000$

Pr TCL : $P(P < \frac{1}{2}) = P\left(\frac{P-\pi}{\sqrt{\frac{\pi(1-\pi)}{8000}}} < \frac{1/2 - \pi}{\sqrt{\frac{\pi(1-\pi)}{8000}}}\right)$

$$= P(U < -3{,}78) = 0{,}00723\%$$

Exercice 3 :

$K \rightsquigarrow B(p)$: $K \rightsquigarrow B(0{,}4)$

1) $K : \begin{cases} 1 & \text{efficace} \\ 0 & \text{non} \end{cases}$

K suit la loi Binominale $B(100; 0{,}4)$

$P(K = k) \quad \binom{n}{k} p^k \cdot q^{n-k}$

$$= \binom{100}{k} p^k \cdot q^{n-k}$$

2, $F_n = \frac{K}{n}$

$n = 100 > 30 \qquad np = 40 > 5 \qquad nq = 60$

$K \rightsquigarrow N(np, npq)$

$E(F_n) = E\left(\frac{K}{n}\right) = \frac{1}{n} E(K) = \frac{np}{n} = p$

$V(F_n) = V\left(\frac{k}{n}\right) = \frac{1}{n^2} V(k) = \frac{npq}{n^2} = \frac{pq}{n}$

$F_n \rightarrow \mathcal{N}(U_f, \sigma_f^2) = \mathcal{N}\left(p, \frac{pq}{n}\right)$

$\mathcal{N}\left(0{,}4 ; \frac{0{,}4 \times 0{,}6}{n}\right)$

3) $P(|F_n - 0{,}4| < t) = 0{,}9$

$P(|F_n - 0{,}4| < t) = 0{,}9$ (90%)

$= P\left(\frac{|F_n - 0{,}4|}{\sqrt{\frac{0{,}4 \times 0{,}6}{100}}} < \frac{t}{\sqrt{\frac{0{,}4 \times 0{,}6}{100}}}\right) = 0{,}9$

$P\left(|U| < \frac{t}{\sqrt{\frac{0{,}4 \times 0{,}6}{100}}}\right) = 0{,}9$

$\frac{t}{\sqrt{\frac{0{,}4 \times 0{,}6}{100}}} = 1{,}645 \Rightarrow t = 1{,}645 \sqrt{\frac{0{,}4 \times 0{,}6}{100}} = 0{,}08$

$P(0{,}32 < F_n < 0{,}48) = 0{,}9$

Ce qui donne comme intervalle 32% - 48%

4) P_2, $F_n = \frac{44}{100} = 0{,}44 \in [32\%, 48\%]$

Exercice 5

1) X_i suit la loi de Bernouilli

$X_i \rightarrow B(p)$

Soit $P(X_i = 1) = 0{,}4$

$P(X_i = 0) = 0{,}6$

$P(X_i = k) = p^k \cdot q^{1-k}$

$X = \sum_{i=1}^{n} X_i$

Soit $n = 250$ jours

X suit loi Binominale

$X \rightsquigarrow B(n,p)$: $K \rightsquigarrow (250\,;\,0,4)$

$P(X=k) = \binom{n}{k} p^k \cdot q^{n-k} = \binom{250}{k} p^k \cdot q^{n-k}$

$E(K) = np$ $\qquad V(K) = npq$

2, $n = 250 > 30$

$np = 100 > 5$

$nq = 150 > 5$

$X \rightsquigarrow N(np, npq) \equiv N(100, 60)$

3, $y = 2000 + 400x$ rémunérat° mensuelle

$y \rightsquigarrow N(\mu_y, \sigma_y^2)$

$U_y = E(y) = E(24000 + 400X)$

$= 24000 + 400\,E(X)$

$= 24000 + 40000 = 64000$

$P(y > 72000) = P(2000 + 400x > 72000)$

$= P(400x > 70000)$

$= P(400X > 48000)$

$= P(X > 120) = P\left(\frac{X-100}{\sqrt{60}} > \frac{120-100}{\sqrt{60}}\right)$

$= P\left(u > \frac{20}{\sqrt{60}}\right) = 1 - \phi(2,586)$ → table 1

$= 1 - 0,9951 = 0,5\%$

$P(y < 60000) = P(24000 + 400X < 60000)$

$= P(400X < 36000)$

$= P(X < 90)$

$= P\left(\frac{X-100}{\sqrt{60}} < \frac{90-100}{\sqrt{60}}\right) = P\left(u < \frac{-10}{\sqrt{60}}\right) = \phi(-1,29)$

$= 1 - \phi(1,29) = 1 - 0,9015 = 9,8\%$

c, $P(60000 < Y < 72000) = P(Y < 72000) - P(Y < 60000)$

$= [1 - P(Y > 72000)] - P(Y < 60000)$

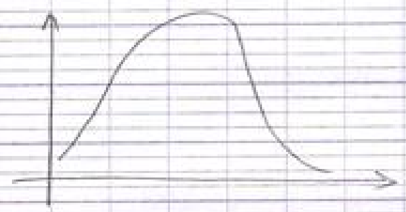
$= 1 - 0,5\% - 9,8\% = 89,7\%$

Exercice 4

$X = e^{m + \sigma u}$

1, Median = 678 €

15,87% des salariés > 1012 €



① $P(X < 678) = 50\%$ médian

② $P(X > 1028) = 15,87\%$

① $P(X < 678) = 50\%$

$\equiv P(\ln X < \ln 678) < 50\%$

$\equiv P(m + \sigma u + \ln 678) < 50\%$

$\equiv P\left(u < \dfrac{\ln 678 - m}{\sigma}\right) < 50\%$

$u \rightsquigarrow N(0,1)$

Donc $\dfrac{\ln 678 - m}{\sigma} = 0 \qquad m = \ln 678$

② $P(X > 1028) = 15,87$

$\equiv P(\ln X > \ln 1012) = 15,87\%$

$\equiv P(m + \sigma u > \ln 1012) = 15,87\%$

$\equiv P\left(u > \dfrac{\ln 1012 - m}{\sigma}\right) = 15,87\%$

$\equiv P\left(u < \dfrac{\ln 1012 - m}{\sigma}\right) = 1 - 15,87\% = 84,13\%$

Table 1

$$\frac{\ln 1012 - m}{\sigma} = 1$$

$$\sigma = \ln 1012 - m = \ln 1012 - \ln 678$$
$$= \ln\left(\frac{1012}{678}\right)$$

$$E(X) = \int_0^{+\infty} x \frac{1}{\sigma x\sqrt{2\pi}} e^{-\frac{(\ln x - m)^2}{2\sigma^2}} dx$$
$$= \int_0^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - m)^2}{2\sigma^2}} dx$$

$$\ln x = y \Rightarrow x = e^y \Rightarrow dx = e^y dy$$
$$= \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-m)^2}{2\sigma^2}} \cdot e^y dy$$
$$= \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2 - 2my + m^2 + 2\sigma^2 y}{2\sigma^2}} dy$$
$$= \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2 - 2my + m^2 - 2\sigma^2 y + \sigma^4 - \sigma^4 + 2m\sigma^2 - 2m\sigma^2}{2\sigma^2}}$$
$$= \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{[y-(m+\sigma^2)]^2}{2\sigma^2}} dy = e^{\sigma^2/2 + m}$$

$$V(X) = E(X^2) - [E(x)]^2$$
$$E(X^2) = \int_0^{+\infty} x^2 \cdot f(x)\, dx = \int_{-\infty}^{+\infty} e^{2y} \frac{1}{e^y \cdot \sigma\sqrt{2\pi}} e^{\frac{(y-m)^2}{2\sigma^2}} e^y \cdot dy$$
$$= e^{2(m+\sigma^2)}$$
$$V(X) = e^{2(m+\sigma^2)} - e^{\sigma^2 + 2m}$$
$$= e^{\sigma^2 + 2m}\left[e^{\sigma^2} - 1\right]$$