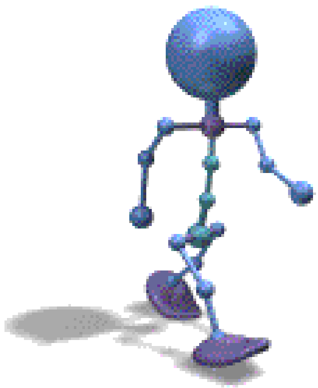
Lundi, 19 Octobre 2015

Optimisation – Licence 3 EF & DU – 2015/2016

Planche de TD N° 3

Développement limité



--- 11H00 → 12H30

Rappel du cours

Soit f une fonction C^n au voisinae de x_0.
Alors f admet un dévelopement limité d′odre n au point x_0 donné par :

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \ldots + \frac{(x-x_0)^n}{n!}f^{(n)}(x_0) + o\big((x - x_0)^n\big)$$

$$f(x) = \sum_{k=0}^{n} \frac{(x - x_0)^k}{k!} f^{(k)}(x_0) + o\big((x - x_0)^n\big)$$

$$o\big((x - x_0)^n\big) = (x - x_0)^n \varepsilon(x) \quad avec \quad \lim_{x \to x_0} \varepsilon(x) = 0$$

Exercice 1

$Soit\ f : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto f(x) = 3xe^{2x+1}$

Ecrire la formule de développement limité à l'ordre 2 de la fonction f au voisinage du point $x^* = 1$.

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + (x-1)^2\varepsilon(x)\ ; avec \quad \lim_{x\to 1}\varepsilon(x) = 0$$

$f(x) = 3xe^{2x+1}$ et $f(1) = 3e^3$

$f'(x) = 3e^{2x+1} + 6xe^{2x+1}$ et $f'(1) = 3e^3 + 6e^3 = 9e^3$

$f''(x) = 6e^{2x+1} + 6e^{2x+1} + 12xe^{2x+1} = 12(x+1)e^{2x+1}$ et $f''(1) = 24e^3$

$$\boxed{f(x) = 3e^3 + 9e^3(x-1) + \frac{24e^3(x-1)^2}{2} + (x-1)^2\varepsilon(x); \quad avec \quad \lim_{x\to 1}\varepsilon(x) = 0}$$

Rappel du cours

$f: \mathbb{R}^n \to \mathbb{R}; x = (x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n)$

$f\ est\ C^2$ au voisinage de $x^* = (x_1^*, \dots, x_n^*)$

La formule du développement d'ordre 2 de f au voisinage de x^* est :

$$
\begin{aligned}
f(x) \quad = \quad & f(x^*) \\
& + \sum_{i=1}^{n} (x_i - x_i^*) \frac{\partial f}{\partial x_i}(x_1^*, \dots, x_n^*) \\
& + \frac{1}{2} \sum_{i,j=1}^{n} (x_i - x_i^*)(x_j - x_j^*) \frac{\partial^2 f}{\partial x_i \partial x_j}(x_1^*, \dots, x_n^*) \\
& + \|(x_1 - x_1^*, \dots, x_n - x_n^*)\|^2 \varepsilon(x_1 - x_1^*, \dots, x_n - x_n^*)
\end{aligned}
$$

$$avec \quad \lim_{(x_1,\dots,x_n) \to (x_1^*,\dots,x_n^*)} \varepsilon(x_1 - x_1^*, \dots, x_n - x_n^*) = 0$$

Exercice 2

$f\ :\ \mathbb{R}\times\,]-1,+\infty[;\ \ (x_1,x_2)\mapsto f(x_1,x_2)=e^{x_1}ln(1+x_2)$

Développement limité d'ordre 2 au voisinage de $(0,0)$

$f(x_1,x_2)=e^{x_1}ln(1+x_2)$ et $f(0,0)=0$

$$\frac{\partial f}{\partial x_1}(x_1,x_2)=e^{x_1}ln(1+x_2)\quad \text{et}\quad \frac{\partial f}{\partial x_1}(0,0)=0$$

$$\frac{\partial f}{\partial x_2}(x_1,x_2)=\frac{e^{x_1}}{1+x_2}\qquad \text{et}\qquad \frac{\partial f}{\partial x_2}(0,0)=1$$

$$\frac{\partial^2 f}{\partial x_1^2}(x_1,x_2)=e^{x_1}ln(1+x_2)\quad \text{et}\quad \frac{\partial^2 f}{\partial x_1^2}(0,0)=0$$

$$\frac{\partial^2 f}{\partial x_2^2}(x_1,x_2)=\frac{-e^{x_1}}{(1+x_2)^2}\quad \text{et}\quad \frac{\partial^2 f}{\partial x_2^2}(0,0)=-1$$

$$\frac{\partial^2 f}{\partial x_2\partial x_1}(x_1,x_2)=\frac{e^{x_1}}{1+x_2}=\frac{\partial^2 f}{\partial x_1\partial x_2}(x_1,x_2)\ \text{et}\ \frac{\partial^2 f}{\partial x_2\partial x_1}(0,0)=1$$

$$f(x) = f(x_1^*, x_1^*)$$
$$+ (x_1 - x_1^*)\frac{\partial f}{\partial x_1}(x_1^*, x_1^*) + (x_1 - x_2^*)\frac{\partial f}{\partial x_2}(x_1^*, x_2^*)$$
$$+ \frac{(x_1 - x_1^*)^2}{2}\frac{\partial^2 f}{\partial x_1^2}(x_1^*, x_2^*) + (x_1 - x_1^*)(x_2 - x_2^*)\frac{\partial^2 f}{\partial x_2 \partial x_1}(x_1^*, x_2^*) + \frac{(x_2 - x_2^*)^2}{2}\frac{\partial^2 f}{\partial x_2^2}(x_1^*, x_2^*)$$
$$+ \|(x_1 - x_1^*, x_2 - x_2^*)\|^2 \varepsilon(x_1 - x_1^*, x_2 - x_2^*)$$

$$avec \quad \lim_{(x_1, x_2) \to (x_1^*, x_2^*)} \varepsilon(x_1 - x_1^*, x_2 - x_2^*) = 0$$

$$f(x) = 0$$
$$+ (x_1 - 0) \times 0 + (x_2 - 0) \times 1$$
$$+ \frac{(x_1 - 0)^2}{2} \times 0 + (x_1 - 0)(x_2 - 0) \times 1 + \frac{(x_2 - 0)^2}{2} \times (-1)$$
$$+ \|(x_1 - 0, x_2 - 0)\|^2 \varepsilon(x_1 - 0, x_2 - 0)$$

$$avec \quad \lim_{(x_1, x_2) \to (0,0)} \varepsilon(x_1 - 0, x_2 - 0) = 0$$

$$f(x_1, x_2) = x_2 + x_1 x_2 - \frac{1}{2}x_2^2 + \|(x_1, x_2)\|^2 \varepsilon(x_1, x_2) \quad avec \quad \lim_{(x_1, x_2) \to (0,0)} \varepsilon(x_1, x_2) = 0$$

Exercice 3

$$f: \mathbb{R}^2_{++} \to \mathbb{R}; x = (x_1, x_2) \mapsto f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha} \quad avec \quad 0 < \alpha < 1$$

Formule de développement d'ordre 2 de f au voisinage de $x^* = (x_1^*, x_2^*)$

$$\begin{aligned} f(x_1, x_2) = {} & f(x_1^*, x_1^*) \\ & + (x_1 - x_1^*) \frac{\partial f}{\partial x_1}(x_1^*, x_1^*) + (x_2 - x_2^*) \frac{\partial f}{\partial x_2}(x_1^*, x_2^*) \\ & + \frac{(x_1 - x_1^*)^2}{2} \frac{\partial^2 f}{\partial x_1^2}(x_1^*, x_2^*) + \frac{(x_1 - x_1^*)(x_2 - x_2^*)}{2} \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_1^*, x_2^*) \\ & + \frac{(x_1 - x_1^*)(x_2 - x_2^*)}{2} \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1^*, x_2^*) + \frac{(x_2 - x_2^*)^2}{2} \frac{\partial^2 f}{\partial x_2^2}(x_1^*, x_2^*) + \\ & + \|(x_1 - x_1^*, x_2 - x_2^*)\|^2 \varepsilon(x_1 - x_1^*, x_2 - x_2^*) \end{aligned}$$

$$avec \quad \lim_{(x_1, x_2) \to (x_1^*, x_2^*)} \varepsilon(x_1 - x_1^*, x_2 - x_2^*) = 0$$

$$f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$$

$$\frac{\partial f}{\partial x_1}(x_1, x_2) = \alpha x_1^{\alpha-1} x_2^{1-\alpha}$$

$$\frac{\partial f}{\partial x_2}(x_1, x_2) = (1-\alpha) x_1^{\alpha} x_2^{-\alpha}$$

$$\frac{\partial^2 f}{\partial x_1^2}(x_1, x_2) = \alpha(\alpha-1) x_1^{\alpha-2} x_2^{1-\alpha}$$

$$\frac{\partial^2 f}{\partial x_2^2}(x_1, x_2) = -\alpha(1-\alpha) x_1^{\alpha} x_2^{-\alpha-1}$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1}(x_1, x_2) = \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} = \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1, x_2)$$

On en déduit que :

$$f(x_1, x_2) = f(x_1^*, x_2^*) + (x_1 - x_1^*)\alpha x_1^{*\,\alpha-1} x_2^{*\,1-\alpha} + (x_2 - x_2^*)(1-\alpha) x_1^{*\,\alpha} x_2^{*\,-\alpha}$$

$$+ \frac{(x_1 - x_1^*)^2}{2}\alpha(\alpha-1) x_1^{*\,\alpha-2} x_2^{*\,1-\alpha}$$

$$+ (x_1 - x_1^*)(x_2 - x_2^*)\alpha(1-\alpha) x_1^{*\,\alpha-1} x_2^{*\,-\alpha}$$

$$+ \frac{(x_2 - x_2^*)^2}{2}\alpha(\alpha-1) x_1^{*\,\alpha} x_2^{*\,-\alpha-1}$$

$$+ \|(x_1 - x_1^*, x_2 - x_2^*)\|^2 \varepsilon(x_1 - x_1^*, x_2 - x_2^*)$$

$$avec \quad \lim_{(x_1, x_2) \to (x_1^*, x_2^*)} \varepsilon(x_1 - x_1^*, x_2 - x_2^*) = 0$$

On en déduit que :

$$\frac{f(x_1,x_2)-f(x_1^*,x_2^*)}{f(x_1^*,x_2^*)} = (x_1-x_1^*)\,\frac{\alpha {x_1^*}^{\alpha-1}{x_2^*}^{1-\alpha}}{{x_1^*}^{\alpha}{x_2^*}^{1-\alpha}} + (x_2-x_2^*) + (x_2-x_2^*)\,\frac{(1-\alpha){x_1^*}^{\alpha}{x_2^*}^{-\alpha}}{{x_1^*}^{\alpha}{x_2^*}^{1-\alpha}}$$

$$+ \frac{(x_1-x_1^*)^2}{2}\frac{\alpha(\alpha-1){x_1^*}^{\alpha-2}{x_2^*}^{1-\alpha}}{{x_1^*}^{\alpha}{x_2^*}^{1-\alpha}}$$

$$+(x_1-x_1^*)(x_2-x_2^*)\,\frac{\alpha(1-\alpha){x_1^*}^{\alpha-1}{x_2^*}^{-\alpha}}{{x_1^*}^{\alpha}{x_2^*}^{1-\alpha}}$$

$$+ \frac{(x_2-x_2^*)^2}{2}\frac{\alpha(\alpha-1){x_1^*}^{\alpha}{x_2^*}^{-\alpha-1}}{{x_1^*}^{\alpha}{x_2^*}^{1-\alpha}}$$

$$+\frac{\|(x_1-x_1^*,x_2-x_2^*)\|^2}{{x_1^*}^{\alpha}{x_2^*}^{1-\alpha}}\,\varepsilon(x_1-x_1^*,x_2-x_2^*)$$

$$avec \quad \lim_{(x_1,x_2)\to(x_1^*,x_2^*)} \varepsilon(x_1-x_1^*,x_2-x_2^*) = 0$$

On en déduit que :

$$\frac{f(x_1,x_2)-f(x_1^*,x_2^*)}{f(x_1^*,x_2^*)} = (x_1-x_1^*)\frac{\alpha}{x_1^*} + (x_2-x_2^*)\frac{(1-\alpha)}{x_2^*}$$

$$+ \frac{(x_1-x_1^*)^2}{2}\frac{\alpha(\alpha-1)}{x_1^{*2}}$$

$$+ (x_1-x_1^*)(x_2-x_2^*)\frac{\alpha(1-\alpha)}{x_1^*x_2^*}$$

$$+ \frac{(x_2-x_2^*)^2}{2}\frac{\alpha(\alpha-1)}{x_2^{*2}}$$

$$+ \frac{\|(x_1-x_1^*,x_2-x_2^*)\|^2}{{x_1^*}^{\alpha}{x_2^*}^{1-\alpha}}\varepsilon(x_1-x_1^*,x_2-x_2^*)$$

$$avec \quad \lim_{(x_1,x_2)\to(x_1^*,x_2^*)} \varepsilon(x_1-x_1^*,x_2-x_2^*) = 0$$

On en déduit que :

$$\frac{f(x_1,x_2)-f(x_1^*,x_2^*)}{f(x_1^*,x_2^*)} = \alpha\,\frac{(x_1-x_1^*)}{x_1^*} + (1-\alpha)\,\frac{(x_2-x_2^*)}{x_2^*}$$

$$+\frac{\alpha(\alpha-1)}{2}\,\frac{(x_1-x_1^*)^2}{{x_1^*}^2}$$

$$-\frac{\alpha(\alpha-1)}{2}\,\frac{(x_1-x_1^*)(x_2-x_2^*)}{x_1^*x_2^*}$$

$$+\frac{\alpha(\alpha-1)}{2}\,\frac{(x_2-x_2^*)^2}{{x_2^*}^2}$$

$$+\frac{\|(x_1-x_1^*,x_2-x_2^*)\|^2}{{x_1^*}^{\alpha}{x_2^*}^{1-\alpha}}\,\varepsilon(x_1-x_1^*,x_2-x_2^*)$$

$$avec \quad \lim_{(x_1,x_2)\to(x_1^*,x_2^*)} \varepsilon(x_1-x_1^*,x_2-x_2^*) = 0$$

On en déduit que :

$$\frac{f(x_1,x_2)-f(x_1^*,x_2^*)}{f(x_1^*,x_2^*)} = \alpha\,\frac{(x_1-x_1^*)}{x_1^*} + (1-\alpha)\,\frac{(x_2-x_2^*)}{x_2^*}$$

$$+\frac{\alpha(\alpha-1)}{2}\left[\frac{x_1-x_1^*}{x_1^*}-\frac{x_2-x_2^*}{x_2^*}\right]^2$$

$$+\frac{\|(x_1-x_1^*,x_2-x_2^*)\|^2}{{x_1^*}^{\alpha}{x_2^*}^{1-\alpha}}\,\varepsilon(x_1-x_1^*,x_2-x_2^*)$$

avec $\lim_{(x_1,x_2)\to(x_1^*,x_2^*)} \varepsilon(x_1-x_1^*,x_2-x_2^*) = 0$

On en déduit que :

$$\frac{f(x_1, x_2) - f(x_1^*, x_2^*)}{f(x_1^*, x_2^*)} \approx \alpha \frac{(x_1 - x_1^*)}{x_1^*} + (1-\alpha) \frac{(x_2 - x_2^*)}{x_2^*} + \frac{\alpha(\alpha-1)}{2} \left[\frac{x_1 - x_1^*}{x_1^*} - \frac{x_2 - x_2^*}{x_2^*}\right]^2$$

Fin

Merci Pour Votre Attention